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Some improvements of HN method with sixth-order covergence

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Abstract: Two families of sixth-order methods are developed by extending a third-order HN method (harmonic mean Newton's method) for finding the real roots of nonlinear equation in R. The convergence analysis is provided to establish their sixth-order of convergence. In terms of computational cost, they require evaluations of only two functions and two first derivatives per iteration. This implies that efficiency index of our methods are 1.565. Our methods are comparable with Newton's method, HN method and others, as we show in some examples. In the end, some improvements of AN method (arithmetic mean Newton's method) were given.

Key words: sixth-order convergence; non linear equation; HN method; iterative method

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具有六阶收敛的HN方法的一些推广

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摘要:通过推广三阶的调和平均牛顿法(HN方法),给出了2类在实数范围内求解非线性方程的六阶方法及其收敛性证明.考虑计算效率,本文方法每步计算2个函数值和2个导数值,效率指数为1.565.将本文方法与牛顿法、HN方法及其他已知方法进行比较,结果表明了本文方法的优越性.最后给出了AN方法(代数平均牛顿方法)的推广.

关键词:六阶收敛;非线性方程;调和平均牛顿法;迭代方法

0 Introduction

Solving nonlinear equation is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x)=0$, where f: $D\subset R\rightarrow R$ for an

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open interval D is a scalar function.

The classical Newton's method for a single nonlinear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is an important and basic method[1], which converges quadratically.

Some modifications of Newton's method with cubic convergence have been developed in[2 - 7]. The Jarratt method[8] is fourth-order convergence. Recently, improvements of Chebyshev-Halley methods with fifth-order convergence is developed in[9], variants of Chebyshev-Halley methods is developed in [10], improvements of the Jarratt method with sixth-order convergence have been developed by Kou Jisheng in[11], by Wang Xiuhua et al in[12] and by Changbum Chun in[13] separately, which improve the local order of convergence of Jarratt method by an additional evaluation of the function[14]. is a sixth-order method, which improves arithmetic mean Newton's method (AN method) in[7].

HN method is a third-order method of A. Y. Özban in[2], which is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)}$$

In this paper, we improve HN method from third-order to sixth-order and only add one first derivative per iteration. In terms of computational cost, it requires the evaluations of only two functions and two first derivatives per iteration. This gives 1.565 as an efficiency index of our methods. Our methods are comparable with Newton's method and other methods. The efficacy of the methods is tested on a number of numerical examples. It is observed that our methods take less number of iterations than taken by Newton's method and other methods. On comparison with the other sixth-order methods, they behave either similarly or better for the examples considered. In section 4, we give improvement of AN method, we can see that S. K. Parhi and D. K. Gupta's method in[14] is a special case of our improvement.

1 The methods and analysis of convergence

Applying method of undetermined cofficient, we consider the following iteration scheme:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)(af'(x_n) + bf'(y_n))}{cf'^2(x_n) + df'(x_n)f'(y_n) + ef'^2(y_n)} \end{aligned} \qquad ①$$

where $a, b, c, d, e \in R$ are constants.

Theorem 1 Assume that function $f \in C^3(D)$ has a simple zero $\partial \in D$. If the initial point x_0 is sufficiently close to α, then the methods defined by ① converge to α with sixth-order under the conditions $c = -\frac{1}{2}(a + b), d = (2a + b), e = \frac{1}{2}(b - a)$.

Proof Considering iteration function of ①

$$F(x) = z - \frac{f(z)(af'(x) + bf'(y))}{cf'^2(x) + df'(x)f'(y) + ef'^2(y)}$$

Where $y = x - \frac{f(x)}{f'(x)}, z = x - \frac{f(x)(f'(x) + f'(y))}{2f'(x)f'(y)}$.

Then ① becomes

$$x_{n+1} = F(x_n)$$

We expand $F(x_n)$ about α, let $e_n = x_n - \alpha$ and $f^{(i)} = f^{(i)}(\alpha)$, then we have

$$F(x_n) = F(\alpha) + F'(\alpha)e_n + \frac{F''(\alpha)}{2!}e_n^2 + \frac{F'''(\alpha)}{3!}e_n^3 + \frac{F^{(4)}(\alpha)}{4!}e_n^4 + \frac{F^{(5)}(\alpha)}{5!}e_n^5 + \frac{F^{(6)}(\alpha)}{6!}e_n^6 + O(e_n^7)$$

Cosidering $f(\alpha) = 0$, after computing we get

$$F(a) = a \qquad F'(a) = 0 \qquad F''(a) = 0$$

$$F'''(a) = \frac{(-a - b + c + d + e)f''}{2(c + d + e)f'}$$

Under the condition $-a - b + c + d + e = 0$, there are

$$c = a + b - d - e \quad F'''(a) = 0$$

$$F^{(4)}(a) = \frac{2(a + 2b - d - 2e)f'f''}{(a + b)f^2}$$

Under the condition $a + 2b - d - 2e = 0$, there are

$$d = 2 + 2b - 2e \qquad F^{(4)}(a) = 0$$

$$F^{(5)}(a) = \frac{5(a - b + 2e)f''^2 f''}{(a + b)f^3}$$

Under the condition $a - b + 2e = 0$, there are

$$e = \frac{1}{2}(b - a) \qquad F^{(5)}(a) = 0$$

$$F^{(6)}(a) = \frac{5f'f''(12af''^2 - 5(a+b)f'f''')}{2(a+b)f'^4}$$

It is clear that $F^{(6)}(a) \neq 0$, then the error equation of ① is thus

$$e_{n+1} = \frac{f'f''(12af''^2 - 5(a+b)f'f''')}{288(a+b)f'^4}e_n^6 + O(e_n^7)$$

After simplifying, we have $c = -\frac{1}{2}(a+b)$, $d = (2a+b)$, $e = \frac{1}{2}(b-a)$. Now it is clear that formula ① converges tricubically under the conditions of theorem. We substitute c, d, e into formula ① and simplify, then we obtain a new family of sixth-order methods as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)} \qquad ②$$

$$x_{n+1} = z_n - \frac{2f(z_n)(af'(x_n) + bf'(y_n))}{-(a+b)f'^2(x_n) + 2(2a+b)f'(x_n)f'(y_n) + (b-a)f'^2(y_n)}$$

where $a, b \in R$ are constants and $a + b \neq 0$. Formula ② converges tricubically.

More generally, we can obtain another new family sixth-order methods as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)} \qquad ③$$

$$x_{n+1} = z_n - \frac{f(z_n)}{(af'(x_n) + bf'(y_n))}H(\mu_n)$$

where a, b are constants, $\mu_n = \frac{f'(y_n)}{f'(x_n)}$ and $H(t)$ represents a real-valued function. Modeled on the proof of Theorem 1, we can get the following conclusion.

Theorem 2 Assume that function $f, H \in C^3(D)$ has a simple zero $\alpha \in D$. If the initial point x_0 is sufficiently close to α then the methods defined by ⑤ converge to α with fourth-order under the conditions $H(1) = a + b$, $H'(1) = -a$, $H''(1) = 3a + b$ and $a + b \neq 0$. The error equation of ③ is

$$e_{n+1} = \frac{f'f''(-5(a+b)f'f''' + 4(f'')^2(15a + 6b + H'''(1)))}{288(a+b)f'^4}e_n^6 + O(e_n^7)$$

In what follows, we give some concrete iterative forms of ③.

Method 1 For the function $H(t)$ defined by

$$H(t) = \frac{7a+3b}{2} - (4a+b)t + \frac{3a+b}{2}t^2 \qquad ④$$

It can easily be seen that the function $H(t)$ of ④ satisfies conditions of Theorem 2. Hence we get a new two-parameter fourth-order family of methods

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{(af'(x_n) + bf'(y_n))} \cdot \left(\frac{(7a+3b)f'^2(x_n) - 2(4a+b)f'(x_n)f'(y_n)}{2f'^2(x_n)} + \frac{(3a+b)f'^2(y_n)}{2f'^2(x_n)} \right) \qquad ⑤$$

Method 2 For the function $H(t)$ defined by

$$H(t) = \frac{(-a^2 + 2ab + b^2) - (a^2 + 4ab + b^2)t}{a + b - (3a+b)t} \qquad ⑥$$

It can easily be seen that the function $H(t)$ of ⑥ satisfies conditions of Theorem 2. Hence we get a new two-parameter fourth-order family of methods

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)} \qquad ⑦$$

$$x_{n+1} = z_n - \frac{f(z_n)}{(af'(x_n) + bf'(y_n))} \cdot \frac{(-a^2 + 2ab + b^2)f'(x_n) - (a^2 + 4ab + b^2)f'(y_n)}{(a+b)f'(x_n) - (3a+b)f'(y_n)}$$

Method 3 For the function $H(t)$ defined by

$$H(t) = \frac{1}{6}(-2 + 13a + 7b + 6t - 3(2 + 5a + b)t^2 + 2(1 + 4a + b)t^3) \qquad ⑧$$

It can easily be seen that the function $H(t)$ of ⑧ satisfies conditions of Theorem 2. Hence we get a new two-parameter fourth-order family of methods

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)} \qquad ⑨$$

$$x_{n+1} = x_n - H\left(\frac{f'(y_n)}{f'(x_n)}\right)\frac{f(z_n)}{(af'(x_n) + bf'(y_n))}$$

Where

$$H\left(\frac{f'(y_n)}{f'(x_n)}\right) = \frac{1}{6}\left(-2 + 13a + 7b + 6\frac{f'(y_n)}{f'(x_n)} - 3(2 + 5a + b)\left(\frac{f'(y_n)}{f'(x_n)}\right)^2 + 2(1 + 4a + b)\left(\frac{f'(y_n)}{f'(x_n)}\right)^3\right)$$

Method 4　Letting $a = 1, b = 0$ in ⑤, for the function $H(t)$ defined by

$$H(t) = \frac{(\alpha + 5\beta - 2\gamma) + \alpha t + \beta t^2}{(-\alpha - 2\beta + \gamma) + (3\alpha + 8\beta - 4\gamma)t + \gamma t^2} \qquad ⑩$$

It can easily be seen that the function $H(t)$ of ⑩ satisfies conditions of Theorem 2. Hence we get a new two-parameter fourth-order family of methods.

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{f(x_n)(f'(x_n) + f'(y_n))}{2f'(x_n)f'(y_n)} \qquad ⑪$$

$$x_{n+1} = x_n - \left(\frac{f(z_n)}{f'(x_n)}\right)\cdot \frac{(\alpha + 5\beta - 2\gamma)f'^2(x_n) + \alpha f'(x_n)f'(y_n) + \beta f'^2(y_n)}{(-\alpha - 2\beta + \gamma)f'^2(x_n) + (3\alpha + 8\beta - 4\gamma)f'(x_n)f'(y_n) + \gamma f'^2(y_n)}$$

where α, β, γ are constants and $\alpha + 3\beta - \gamma \neq 0$.

In terms of computational cost, it requires evaluations of only two functions and two first derivatives per iteration. We consider the definition of efficiency index [15] as $p^{\frac{1}{\omega}}$, where p is the order of the method and ω is the number of function evaluations per iteration required by the method. If we assume that all the evaluations have the same cost as function one, we have that the present methods have the efficiency indexes which equal to $6^{\frac{1}{4}} = 1.565$, which are better than the ones of $3^{\frac{1}{3}} = 1.442$ in [2 - 7], $5^{\frac{1}{4}} = 1.495$ in [9] and Newton's method $2^{\frac{1}{2}} = 1.414$.

2　Numerical results and conclusions

In this section, we present the results of some numerical tests to compare the efficiency of the methods. We employed CN method in [1], HN method in [2], KM in [11], CM1 ($\alpha = 1$) and CM2 ($\alpha = -1$) in [13], VSHM ($\alpha = 1, \gamma = 1$) method in [10] and new methods N1 ($a = 1, b = 1$ in ②), N2 ($a = 1, b = -3$ in ⑤), N3 ($a = 1, b = -3$ in ⑦), N4 ($a = 1, b = -3$ in ⑨), N5 ($\alpha = 0, \beta = 1, \gamma = 0$ in ⑪).

Numerical computations reported here have been carried out in a Mathematica 4.0 environment. The stoping criterion has been taken as $|f(x_{n+1})| + |x_{n+1} - \alpha| < 10^{-14}$. We can see the computing results in Table 1.

The test functions of $f(x)$ are as follows:

$$f_1(x) = \prod_{m=0}^{4}(x - (1 + 0.1m)) \qquad \alpha = 1$$

$$f_2(x) = x^3 + 4x^2 - 10$$

$$\alpha \approx 1.365\,230\,013\,414\,097$$

$$f_3(x) = \cos(x) - x$$

$$\alpha \approx 0.739\,085\,133\,215\,160\,67$$

$$f_4(x) = \sin^2(x) - x^2 + 1$$

$$\alpha \approx 1.404\,491\,648\,215\,341$$

$$f_5(x) = e^{x^2+7x-30} - 1 \qquad \alpha = 3$$

$$f_6(x) = x.e^{x^2} - \sin^2 x + 3\cos x + 5$$

$$\alpha \approx -1.207\,647\,827\,130\,919$$

Table 1　Comparison of various six-order methods, HN method and Newton's method

$f(x)$	x_0	NOFE										
		NM	HN	KM	CM1	CM2	VSHM	N1	N2	N3	N4	N5
$f_1(x)$	-0.5	32	27	NC	NC	27	20	24	24	24	24	24
	0	28	24	32	NC	32	20	20	20	20	20	20
$f_2(x)$	-0.5	30	126	NC	28	NC	NC	76	76	76	16	40
	2.0	10	9	8	8	8	8	8	8	8	8	8
$f_3(x)$	-0.9	14	108	36	44	12	NC	20	20	20	16	28
	1.0	8	9	8	8	8	8	8	8	8	8	8
$f_4(x)$	2.0	10	9	8	8	8	8	8	8	8	8	8
$f_5(x)$	3.5	24	21	20	20	20	20	16	16	16	16	20
	4.0	39	33	20	20	20	NC	24	24	24	24	28
$f_6(x)$	1.0	14	15	NC	NC	NC	NC	12	12	12	12	12
	-2.0	16	15	12	12	12	NC	12	12	12	12	12

In Table 1, $f(x)$ expresses test function, x_0 expresses original iteration value, NOFE expresses the number of function evaluations. NC in Table 1 implies that the method does not converge. Here CN method is second-order, HN method is third-order, the other methods are sixth-order. The results show that the proposed methods have some more advantages than the others. The KM, CM1, CM2 and VSHM methods have sensitivities to the original iteration value and they don't converge (NC) to the zero for some test function. The new methods have iteration stabilities to original iteration value and behave better than the others in most situations.

3 Improvements of AN method

Applying the methods in this paper, we can obtain improvements of AN method as follows, it is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \qquad ⑫$$
$$x_{n+1} = z_n - \frac{2f(z_n)(af'(x_n) + bf'(y_n))}{-(a+b)f'^2(x_n) + 2(2a+b)f'(x_n)f'(y_n) + (b-a)f'^2(y_n)}$$

where $a, b \in R$ are constants and $a + b \neq 0$. Formula ⑫ converges tricubically, the error equation is

$$e_{n+1} = -\frac{f''(3f''^2 + f'f''')(3(-3a+b)f''^2 + 5(a+b)f'f''')}{288(a+b)f'^5} \cdot e_n^6 + O(e_n^7)$$

When $a = 1, b = 1$, formula ⑫ becomes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$z_n = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \qquad ⑬$$
$$x_{n+1} = z_n - \frac{f'(x_n) + f'(y_n)}{-f'(x_n) + 3f'(y_n)} \frac{f(z_n)}{f'(x_n)}$$

⑬ is S. K. Parhi and D. K. Gupta's method in [14]. It is clear that S. K. Parhi and D. K. Gupta's method in [14] is special case of formula ⑫.

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